1. Let $A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

   (a) Use Gauss-Jordan to find the inverse of A.

   (b) Write A^{-1} as a product of elimination matrices.

   (c) Find the determinant of each elimination matrix, and the determinants of A^{-1} and A.

   (d) Does A preserve or reverse orientation?

   (e) What effect does the linear transformation A have on the volume of a region?

   (f) Write A as a product of elimination matrices.

   Solution:

   (a) In the Gauss-Jordan algorithm we start with the matrix $\begin{bmatrix} A & | & I \end{bmatrix}$:

$$\begin{bmatrix} A & | & I \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

   We have a pivot (-1) in Row 1 Column 1, so we add Row 1 to Row 3 to get

$$\begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 & 1 \end{bmatrix}.$$

   Columns 1 and 2 are now clear, so we move to the pivot (3) in Row 3 Column 3, and we subtract (1/3)(Row 3) from Row 1 to get

$$\begin{bmatrix} -1 & 0 & 0 & 2/3 & 0 & -1/3 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 & 1 \end{bmatrix}.$$

   Now we multiply Row 1 by -1 and Row 3 by 1/3 to get

$$\begin{bmatrix} 1 & 0 & 0 & -2/3 & 0 & 1/3 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 0 & 1/3 \end{bmatrix}.$$

   since we now have I_3 in columns 1 to 3,

$$A^{-1} = \begin{bmatrix} -2/3 & 0 & 1/3 \\ 0 & 1 & 0 \\ 1/3 & 0 & 1/3 \end{bmatrix}.$$

(b) The first row operation was adding Row 1 to Row 3, which corresponds to multiplying on the left by the elimination matrix $E_{3,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

The second row operation was subtracting (1/3)(Row 3) from Row 1, which corresponds to multiplying on the left by the elimination matrix $E_{1,3} = \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The third row operation was multiplying Row 1 by -1, which corresponds to multiplying on the left by the elimination matrix $E_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The fourth row operation was multiplying Row 3 by 1/3, which corresponds to multiplying on the left by the elimination matrix $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$.

So

$$A^{-1} = E_3E_1E_{1,3}E_{3,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

(c)

$$\det E_3 = 1/3, \det E_1 = -1, \det E_{1,3} = 1, \text{ and } \det E_{3,1} = 1.$$

Thus $\det A^{-1} = (1/3)(-1)(1)(1) = -1/3$. Since $\det A$ and $\det A^{-1}$ are reciprocals, $\det A = -3$.

(d) Since $\det A < 0$, A reverses orientation.

(e) Since $|\det A| = 3$, A scales the volume of a region by a factor of 3.

(f)

$$A = E_{3,1}^{-1}E_{1,3}^{-1}E_3^{-1}E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

2. Let $A = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 5 & 1 \\ 2 & 3 & 1 \\ 6 & 8 & 2 \\ -8 & -7 & 1 \end{bmatrix}$.

(a) Find the RREF for A.

(b) Let $\mathbf{b} = (b_1, b_2, b_3, b_4, b_5)$. Under what conditions will $A\mathbf{x} = \mathbf{b}$ have a solution?

(c) Let $\mathbf{b} = (1, 1, 0, 1, -5)$. Find all solutions to $A\mathbf{x} = \mathbf{b}$.

(d) What is the rank of A?

(e) Find the dimension for each of the four fundamental subspaces for A.

(f) Find a basis for each of $C(A)$, $C(A^T)$, and $N(A)$.

Solution:

(a) Beginning with $A = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 5 & 1 \\ 2 & 3 & 1 \\ 6 & 8 & 2 \\ -8 & -7 & 1 \end{bmatrix}$, we have a pivot (2) in Row 1 Column 1, so we do R2-2R1, R3-R1, R4-3R1, and R5+4R1 to obtain

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

We have a pivot (1) in row 2 column 2, so we do R1-2R2, R3-R2, R4-2R2, and R5-R2 to get

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now we divide row 1 by 2 to obtain the RREF:

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Let $\mathbf{b} = (b_1, b_2, b_3, b_4, b_5)$. Applying the same row operations to $\mathbf{b}$ as to A, we get

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \to \begin{bmatrix} b_1 \\ -2b_1 + b_2 \\ -b_1 + b_3 \\ -3b_1 + b_4 \\ 4b_1 + b_5 \end{bmatrix} \to \begin{bmatrix} (5b_1 - 2b_2)/2 \\ -2b_1 + b_2 \\ b_1 - b_2 + b_3 \\ b_1 - 2b_2 + b_4 \\ 6b_1 - b_2 + b_5 \end{bmatrix}.$$

Thus $A\mathbf{x} = \mathbf{b}$ if and only if

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} (5b_1 - 2b_2)/2 \\ -2b_1 + b_2 \\ b_1 - b_2 + b_3 \\ b_1 - 2b_2 + b_4 \\ 6b_1 - b_2 + b_5 \end{bmatrix}$$

has a solution. This is possible if and only if $b_1 - b_2 + b_3 = 0$, $b_1 - 2b_2 + b_4 = 0$, and $6b_1 - b_2 + b_5 = 0$.

(In other words, $b_3 = -b_1 + b_2$, $b_4 = -b_1 + 2b_2$, and $b_5 = -6b_1 + b_2$: The components of $\mathbf{b}$ must have the same dependence relations as the rows of A.)

(c) Let $\mathbf{b} = (1, 1, 0, 1, -5)$. Then by Part (b), $A\mathbf{x} = \mathbf{b}$ if and only if

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3/2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note that if $\mathbf{x} = (x_1, x_2, x_3)$ is a solution, the variable x_3 is a free variable while the variables x_1 and x_2 are pivot variables. Thus we can obtain a particular solution by fixing x_3. Set $x_3 = 0$ to get the particular solution $\mathbf{x}_p = (3/2, -1, 0)$.

For the full solution, from the RREF we see that $N(A)$ is 1-dimensional, with basis vector $\mathbf{x}_n = (1, -1, 1)$.

Thus $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{x}$ is of the form

$$\mathbf{x} = \mathbf{x}_p + c\mathbf{x}_n = (3/2, -1, 0) + c(1, -1, 1)$$

for some $c \in \mathbb{R}$.

(d) The rank of A is 2.

(e) $\dim C(A) = r = 2$, $\dim C(A^T) = r = 2$, $\dim N(A) = n - r = 3 - 2 = 1$, and $\dim N(A^T) = m - r = 5 - 2 = 3$.

(f) A basis for $C(A)$ is given by columns 1 and 2 of A: $\{(2, 4, 2, 6, -8), (2, 5, 3, 8, -7)\}$.
A basis for $C(A^T)$ is given by rows 1 and 2 of A: $\{(2, 2, 0), (4, 5, 1)\}$.
A basis for $N(A)$ can be read off the RREF: $\{(1, -1, 1)\}$.

3. (a) State the definition of linear independence.
   (b) State the definition of null space of a matrix.
   (c) Prove that if a matrix has a nonzero vector in its null space, then its columns are not linearly independent.

Solution:

(a) A set of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is linearly independent if for real numbers $c_1, \ldots, c_n$,

$$c_1\mathbf{x}_1 + \cdots c_n\mathbf{x}_n = \mathbf{0}$$

if and only if $c_i = 0$ for all i.

(b) The null space of a matrix A is the set of vectors $\{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$.

(c) Suppose $\mathbf{v} = (v_1, \ldots, v_n) \neq \mathbf{0}$ and $\mathbf{v} \in N(A)$, i.e. $A\mathbf{v} = \mathbf{0}$. Let $\mathbf{x}_i$ be the vector in column i of A. Then $\mathbf{0} = A\mathbf{v} = v_1\mathbf{x}_1 + \cdots + v_n\mathbf{x}_n$.
Since $v_1\mathbf{x}_1 + \cdots + v_n\mathbf{x}_n = \mathbf{0}$ and not all coefficients v_i are equal to 0, the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is not linearly independent, i.e. the columns of A are not linearly independent.

4. Let $\mathbf{v} = (1, 1, -1)$.

   (a) Let P be the matrix such that $P\mathbf{x}$ is the projection of $\mathbf{x}$ onto $\mathbf{v}$ for any $\mathbf{x} \in \mathbb{R}^3$. Find P.

   (b) Let $\mathbf{w} = (1, 2, 0)$. Find the projection of $\mathbf{w}$ onto $\mathbf{v}$.

   (c) Let $V = \text{span}\{\mathbf{v}, \mathbf{w}\}$. Find an orthonormal basis for V.

   (d) Find the projection of the vector $\mathbf{x} = (-1, 5, 1)$ onto V.

   (e) Find a QR decomposition for the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 0 \end{bmatrix}$.

   Solution:

   (a) $P = \mathbf{v}(\mathbf{v}^T\mathbf{v})^{-1}\mathbf{v}^T$. Since $\mathbf{v}^T\mathbf{v} = 3$,

$$P = \frac{\mathbf{v}\mathbf{v}^T}{3} = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

   (b)

$$\text{proj}_{\mathbf{v}}\mathbf{w} = P\mathbf{w} = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

   (c) We use Gram-Schmidt. One basis vector will be $\mathbf{q}_1 = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3}}(1, 1, -1)$. A second basis vector orthogonal to the first is given by

$$\mathbf{w} - \text{proj}_{\mathbf{v}}\mathbf{w} = (1, 2, 0) - (1, 1, -1) = (0, 1, 1).$$

   After normalizing this vector we obtain an orthonormal basis for V, which is

$$\{\mathbf{q}_1, \mathbf{q}_2\} = \left\{\frac{1}{\sqrt{3}}(1, 1, -1), \frac{1}{\sqrt{2}}(0, 1, 1)\right\}.$$

   (d) The projection of $\mathbf{x}$ onto V is the sum of the projections onto $\mathbf{q}_1$ and $\mathbf{q}_2$.

$$\begin{aligned}
&(\mathbf{q}_1^T(-1, 5, 1))\mathbf{q}_1 + (\mathbf{q}_2^T(1, 5, 1))\mathbf{q}_2 \\
&\qquad = \frac{1}{\sqrt{3}}(1, 1, -1)^T(-1, 5, 1)\left(\frac{1}{\sqrt{3}}(1, 1, -1)\right) + \frac{1}{\sqrt{2}}(0, 1, 1)^T(1, 5, 1)\left(\frac{1}{\sqrt{2}}(0, 1, 1)\right) \\
&\qquad = \sqrt{3}\left(\frac{1}{\sqrt{3}}(1, 1, -1)\right) + 3\sqrt{2}\left(\frac{1}{\sqrt{2}}(0, 1, 1)\right) \\
&\qquad = (1, 1, -1) + (0, 3, 3) \\
&\qquad = (1, 4, 2).
\end{aligned}$$

(e)

$$\begin{aligned}
\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 0 \end{bmatrix} &= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T(1,1,-1) & \mathbf{q}_1^T(1,2,0) \\ \mathbf{q}_2^T(1,1,-1) & \mathbf{q}_2^T(1,2,0) \end{bmatrix} \\
&= \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \\ -1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{2} \end{bmatrix} \\
&= QR
\end{aligned}$$

5. Let $A = \begin{bmatrix} -1 & 1.5 \\ -3 & 3.5 \end{bmatrix}$.

(a) Find the characteristic polynomial for A and use it to verify that the eigenvalues of A are 2 and 1/2.

(b) Verify that $(1, 2)$ and $(1, 1)$ are eigenvectors for A.

(c) Use Parts (a) and (b) to find a diagonalization for A.

(d) Find a diagonalization for A^{25}.

(e) Write the vectors $\mathbf{u} = (3, 6)$ and $\mathbf{v} = (2, 5)$ each as a linear combination of the eigenvectors of A.

(f) Find $A^{25}\mathbf{u}$ and $A^{25}\mathbf{v}$. Explain why these vectors are almost identical.

(g) Let $B = \begin{bmatrix} 11 & -3 \\ 4 & 7 \end{bmatrix}$ and suppose $C = BAB^{-1}$. Find the eigenvalues and eigenvectors of C. What is $\det C$?

Solution:

(a) The characteristic polynomial for A is $(-1 - \lambda)(3.5 - \lambda) - (1.5)(-3) = \lambda^2 - 2.5\lambda + 1$.
Since $2^2 - (2.5)(2) + 1 = 0$ and $(1/2)^2 - (2.5)(1/2) + 1 = 0$, the eigenvalues of A are 2 and 1/2.

(b)
$$\begin{bmatrix} -1 & 1.5 \\ -3 & 3.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$
$$\begin{bmatrix} -1 & 1.5 \\ -3 & 3.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} .5 \\ .5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(c)
$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

(d)
$$A^{25} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^{25} & 0 \\ 0 & (1/2)^{25} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

(e) $\mathbf{u} = (3, 6) = 3(1, 2) + 0(1, 1)$, and $\mathbf{v} = (2, 5) = 3(1, 2) - 1(1, 1)$.

(f) $A^{25}\mathbf{u} = 2^{25}(3)(1, 2) = (3 \cdot 2^{25}, 3 \cdot 2^{26})$.
$A^{25}\mathbf{v} = 2^{25}(3)(1, 2) - (1/2)^{25}(1, 1) = (3 \cdot 2^{25} - (1/2)^{25}, 3 \cdot 2^{26} - (1/2)^{25})$.
These vectors are almost identical because the coefficient of the leading eigenvector is the same.

(g) The eigenvalues of C are 2 and 1/2, and the corresponding eigenvectors are
$$\mathbf{x}_1 = B(1, 2) = (5, 18) \text{ and } \mathbf{x}_2 = B(1, 1) = (8, 11).$$
$\det C = \det(BAB^{-1}) = \det B \det A \det B^{-1} = \det B \det A \frac{1}{\det B} = \det A = 1$.

6. (a) State the Spectral theorem.

   (b) State the definition of an orthogonal matrix.

   (c) Suppose S is an $n \times n$ symmetric, real matrix with n positive eigenvalues. Use the Spectral theorem to prove that S can be written as $A^T A$ for some matrix A.

Solution:

(a) If S is an $n \times n$ real, symmetric matrix, then S has n real eigenvalues, and can be diagonalized as $S = Q\Lambda Q^{-1} = Q\Lambda Q^T$, where Q is an orthogonal matrix of eigenvectors and Λ is a diagonal matrix with the corresponding eigenvalues on the diagonal.

(b) A matrix is orthogonal if it is square and the columns are orthonormal, i.e. they are an orthogonal set of unit vectors.

(c) Suppose S is an $n \times n$ symmetric, real matrix with n positive eigenvalues. By the Spectral theorem, $S = Q\Lambda Q^T$, where Q is orthogonal and Λ is a diagonal matrix of eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\sqrt{\Lambda}$ be the diagonal matrix with $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$ on the diagonal. $\Lambda = \sqrt{\Lambda}\sqrt{\Lambda}$, so

$$S = Q\sqrt{\Lambda}\sqrt{\Lambda}Q^T.$$

Since $\sqrt{\Lambda}$ is diagonal, it is symmetric, so

$$S = Q\sqrt{\Lambda}\sqrt{\Lambda}^T Q^T = (Q\sqrt{\Lambda})(Q\sqrt{\Lambda})^T = A^T A,$$

where $A = (Q\sqrt{\Lambda})^T$.

7. Let $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \\ 0 & 0 \end{bmatrix}$.

(a) Find the full format singular value decomposition for A.

(b) What is the rank r of A?

(c) Find the rank r format singular value decomposition for A.

(d) Use the singular value decomposition to write A as a sum of r rank 1 matrices.

(e) If $\mathbf{x}$ is a unit vector, what are the maximum and minimum possible values of $\|A\mathbf{x}\|$? For which vectors $\mathbf{x}$ are these values achieved?

Solution:

(a) First we calculate the singular values and right singular vectors by finding eigenvalues and eigenvectors of A^TA:

$$A^TA = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

A^TA has eigenvalues 4 and 1, with eigenvectors $(1,0)$ and $(0,1)$.
Thus A has singular values $\sigma_1 = \sqrt{4} = 2$ and $\sigma_2 = \sqrt{1} = 1$, with singular vectors $\mathbf{v}_1 = (1,0)$ and $\mathbf{v}_2 = (0,1)$.
Similarly,

$$AA^T = \begin{bmatrix} 0 & 1 \\ -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

AA^T has eigenvalues 1, 4, and 0 with eigenvectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.
Thus A has singular values $\sigma_1 = \sqrt{4} = 2$ and $\sigma_2 = \sqrt{1} = 1$, with singular vectors $\mathbf{u}_1 = (0,1,0)$, $\mathbf{u}_2 = (1,0,0)$, and $\mathbf{u}_3 = (0,0,1)$.
To check that the signs for U and V are compatible, we check that $A\mathbf{v}_1 = \sigma_1\mathbf{u}_1$:

$$\begin{bmatrix} 0 & 1 \\ -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

For this equation to be satisfied, we use $\mathbf{u}_1 = (0,-1,0)$.
We also check $A\mathbf{v}_2 = \sigma_2\mathbf{u}_2$.

$$\begin{bmatrix} 0 & 1 \\ -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The signs are compatible here, so we keep $\mathbf{u}_2 = (1,0,0)$.

Thus the full format SVD is

$$\begin{bmatrix} 0 & 1 \\ -2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U\Sigma V^T.$$

(b) A has rank $r = 2$.

(c) The rank 2 format singular value decomposition for A is

$$\begin{bmatrix} 0 & 1 \\ -2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U_2\Sigma_2 V_2^T.$$

(d)

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ -2 & 0 \\ 0 & 0 \end{bmatrix} &= \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T \\ &= 2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ -2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

(e) If $\mathbf{x}$ is a unit vector, the maximum possible value of $\|A\mathbf{x}\|$ is $\sigma_1 = 2$, achieved when $\mathbf{x} = \pm\mathbf{v}_1$:

$$\|A\mathbf{v}_1\| = \|(0, -2, 0)\| = 2.$$

If $\mathbf{x}$ is a unit vector, the minimum possible value of $\|A\mathbf{x}\|$ is $\sigma_2 = 1$, achieved when $\mathbf{x} = \pm\mathbf{v}_2$:

$$\|A\mathbf{v}_2\| = \|(1, 0, 0)\| = 1.$$